# Trigonometric form of a complex number (polar form)

**To remind you:**

Complex number is : $z = x + yi$

x is a real part, y is imaginary part of a complex number, and $i$ is imagirary unit : $i = \sqrt{-1}$ $(i^2 = -1)$

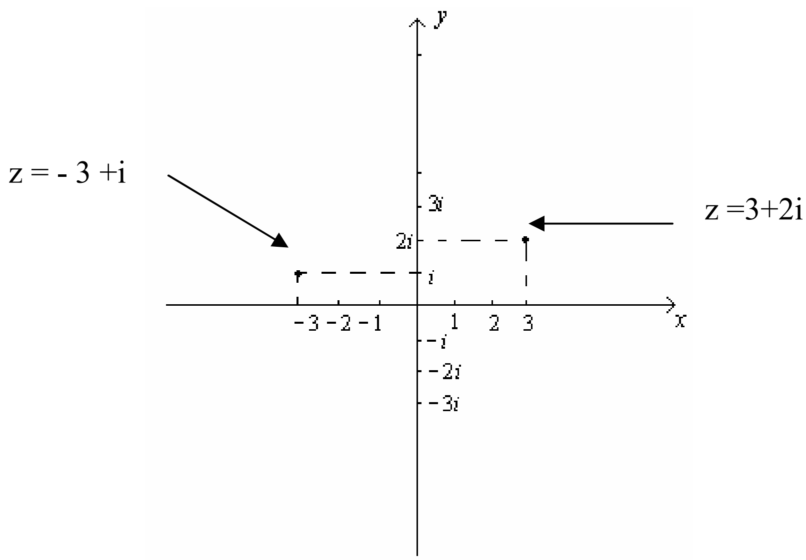
Two complex number $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are equal if $x_1 = x_2$ and $y_1 = y_2$

For $z = x + yi$ , number $\bar{z} = x - yi$ is a complex conjugate number.

Module of complex number $z = x + yi$ is $|z| = \sqrt{x^2 + y^2}$

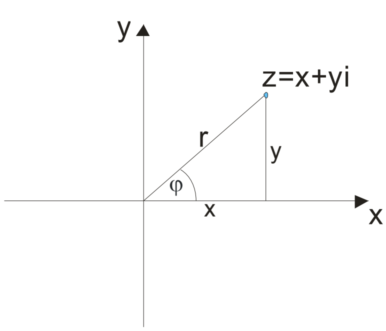
Complex numbers are presented in the complex plane, where x-line is real line and y-line is imaginary line:

**Example:**




If we have complex number $z = x + yi$ then the real part can write as: $x = r\cos\varphi$ ,and imaginary $y = r\sin\varphi$

This we can see from the picture:




$$\sin\varphi = \frac{y}{r} \Rightarrow y = r\sin\varphi$$

$$\cos\varphi = \frac{x}{r} \Rightarrow x = r\cos\varphi$$

$$tg\varphi = \frac{y}{x} \Rightarrow \varphi = arctg\frac{y}{x}$$

Therefore, the complex number is:

$$z = r\cos\varphi + r\sin\varphi i, or$$
$$z = r(\cos\varphi + i\sin\varphi)$$

This form is called trigonometric. Here , r is module $r = \sqrt{x^2 + y^2}$ ;angle φ is the argument of the complex number. How are sinx and cosx periodic functions complex number we can write as:

$$z = r(\cos(\varphi + 2k\pi) + i\sin(\varphi + 2k\pi))$$
$$k \in Z$$

**Example**: Make the following complex numbers in a trigonometrical form:

a) $z = 1 + i$
b) $z = 1 + i\sqrt{3}$
c) $z = -1$
d) $z = i$

**Solution:**

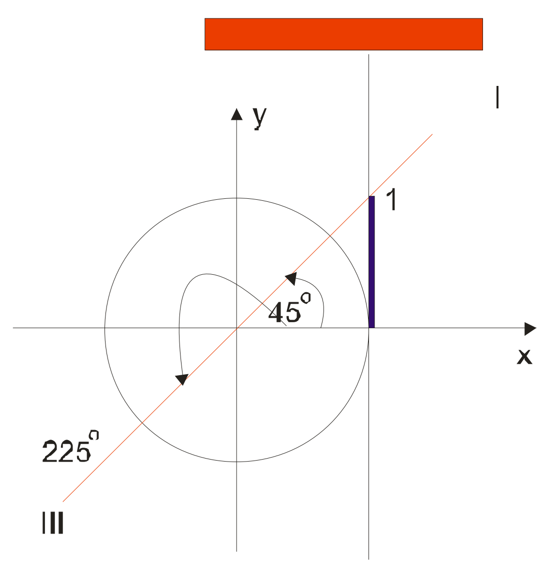
a) $z = 1 + i$

What we do?

First, determine x and y, then find $r = \sqrt{x^2 + y^2}$ , then $tg\varphi = \frac{y}{x}$ and replace that to trigonometriski form:

$$z = r(\cos\varphi + i\sin\varphi)$$

So: $x = 1, y = 1 \quad r = \sqrt{1^2 + 1^2} = \sqrt{2}$




$$tg\varphi = \frac{y}{x}$$

$$tg\varphi = \frac{1}{1}$$

$$tg\varphi = 1 \Rightarrow \varphi = 45^o = \frac{\pi}{4}$$

$$z = r(\cos\varphi + i\sin\varphi)$$

$z = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ solution

b) $z = 1 + i\sqrt{3}$

$x = 1$

$y = \sqrt{3}$ $\Rightarrow r = \sqrt{x^2 + y^2} = \sqrt{1+3} = \sqrt{4} = 2$

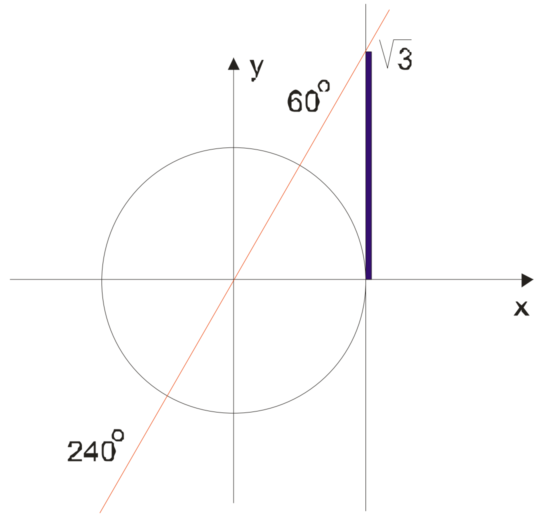
$tg\varphi = \frac{y}{x} = \frac{\sqrt{3}}{1}$

$tg\varphi = \sqrt{3}$

$\varphi = 60^o = \frac{\pi}{3}$

$z = r(\cos\varphi + i\sin\varphi)$

$z = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$




c) $z = -1$ Take heed: This can record as: $z = -1 + 0i$

So :

$x = -1, y = 0$

$tg\varphi = \frac{y}{x} = \frac{0}{-1} = 0$

$\varphi = 180^o$

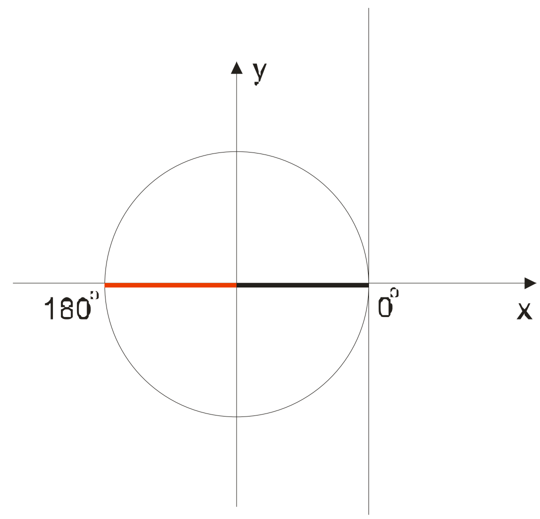
$\varphi = \pi$

$r = \sqrt{(-1)^2 + 0^2} = 1$

$z = r(\cos\varphi = i\sin\varphi)$

$z = 1 \cdot (\cos\pi + i\sin\pi)$

$z = \cos\pi + i\sin\pi$




d) $z = i$ or $z = 0 + 1i \Rightarrow x = 0, y = 1$

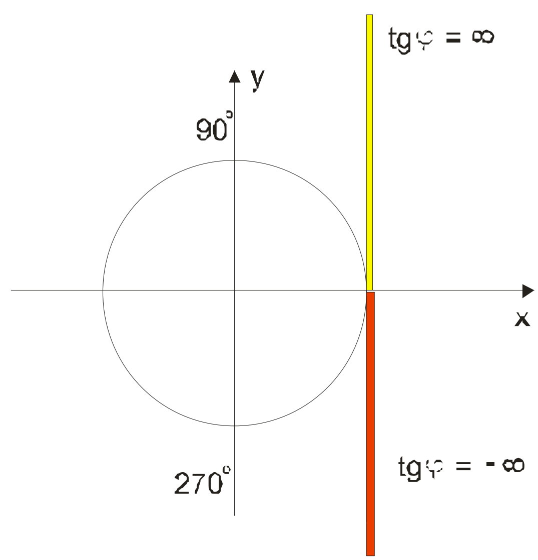
$r = \sqrt{0^2 + 1^2} = 1$

$tg\varphi = \frac{1}{0} = \infty$

$\varphi = \frac{\pi}{2}$

$z = r(\cos\varphi + i\sin\varphi)$

$z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$



Often, in solving tasks we use **Euler formula:**

$$e^{xi} = \cos x + i \sin x$$

**Example:** Write numbers through Euler formula.

a) 1

b) i

c) -2

**Solution:**

**Tip:** Here always add periodicity!

a) $z = 1 \longrightarrow x = 1, y = 0$

$r = \sqrt{1^2 + 0^2} = 1$

$tg\varphi = \frac{y}{x} = 0 \Rightarrow \varphi = 0^o$

$z = r(\cos(\varphi + 2k\pi) + i\sin(\varphi + 2k\pi))$

$z = 1 \cdot (\cos(0 + 2k\pi) + i\sin(0 + 2k\pi))$

So: $1 = \cos 2k\pi + i \sin 2k\pi$ , or

$1 = e^{2k\pi i}$

$k \in Z$

b) $z = i \quad \Rightarrow \quad z = 0 + 1i \quad \Rightarrow \quad x = 0, y = 1$

$r = \sqrt{x^2 + y^2} = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$

$tg\varphi = \frac{y}{x} = \frac{1}{0} = \infty \Rightarrow \varphi = \frac{\pi}{2}$

$z = r(\cos(\varphi + 2k\pi) + i\sin(\varphi + 2k\pi))$

$z = \cos(\frac{\pi}{2} + 2k\pi) + i\sin(\frac{\pi}{2} + 2k\pi)$

So: $i = \cos(\frac{\pi}{2} + 2k\pi) + i\sin(\frac{\pi}{2} + 2k\pi)$

$i = e^{(\frac{\pi}{2} + 2k\pi)i}$

$k \in Z$

c) $z = -2 = 2 \cdot (-1) =$

-1 ⟶ we find in the previous example:

$$-1 = \cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)$$

$$-2 = 2[\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)]$$

So:

$$-2 = 2e^{(\pi + 2k\pi)i}$$

$$-2 = 2e^{(2k+1)\pi i}$$

$$k \in Z$$

**Professors often like to ask the children to find values** $i^i$

When you know Euler's record, it is not difficult.

In a previous example, we find:

$$i = e^{(\frac{\pi}{2} + 2k\pi)i}$$

$$k \in Z$$

Then: $i^i = (e^{(\frac{\pi}{2} + 2k\pi)i})^i$

$$i^i = e^{(\frac{\pi}{2} + 2k\pi)i^2}$$

$$i^i = e^{-(\frac{\pi}{2} + 2k\pi)}$$

$$k \in Z$$

$\boxed{i^i = e^{-\frac{\pi}{2}}}$ is for k = 0

**Multiplication of complex numbers in trigonometric form**

If we have two complex number in the trigonometric form:

$$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$$
$$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$$

then:

$$z_1 \cdot z_2 = r_1 \cdot r_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)]$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)]$$

**Examples: We have**

$$z_1 = 4\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$$
$$z_2 = \frac{\sqrt{2}}{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$$

**find:** a) $z_1 \cdot z_2$ b) $\frac{z_1}{z_2}$

Solution:

a)

$$z_1 \cdot z_2 = 4\sqrt{2} \cdot \frac{\sqrt{2}}{2}\left[\cos(\frac{\pi}{4} + \frac{3\pi}{4}) + i\sin(\frac{\pi}{4} + \frac{3\pi}{4})\right]$$
$$= 4[\cos\pi + i\sin\pi]$$
$$= 4[-1 + 0]$$
$$= -4$$

b)

$$\frac{z_1}{z_2} = \frac{4\sqrt{2}}{\frac{\sqrt{2}}{2}}\left[\cos(\frac{\pi}{4} - \frac{3\pi}{4}) + i\sin(\frac{\pi}{4} - \frac{3\pi}{4})\right]$$
$$= 8\left[\cos(-\frac{\pi}{2}) + i\sin\frac{\pi}{2}\right]$$
$$= 8[0 - i \cdot 1]$$
$$= -8i$$

**De Moivre's formula**

If we have $z = r(\cos\varphi + i\sin\varphi)$ then $\boxed{z^n = r^n(\cos n\varphi + i\sin n\varphi)}$

If the complex has a number has a module 1, ie. if r = 1 then:

$$z = \cos\varphi + i\sin\varphi$$
$$z^n = \cos n\varphi + i\sin n\varphi \rightarrow \text{De Moivre's formula}$$

**Examples :** a) Find $z^6$ if $z = 2(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18})$

b) Find $z^{20}$ if $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

**Solution:**

a)

$$z = 2(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18})$$

$$z^6 = 2^6(\cos 6\cdot\frac{\pi}{18} + i\sin 6\cdot\frac{\pi}{18})$$

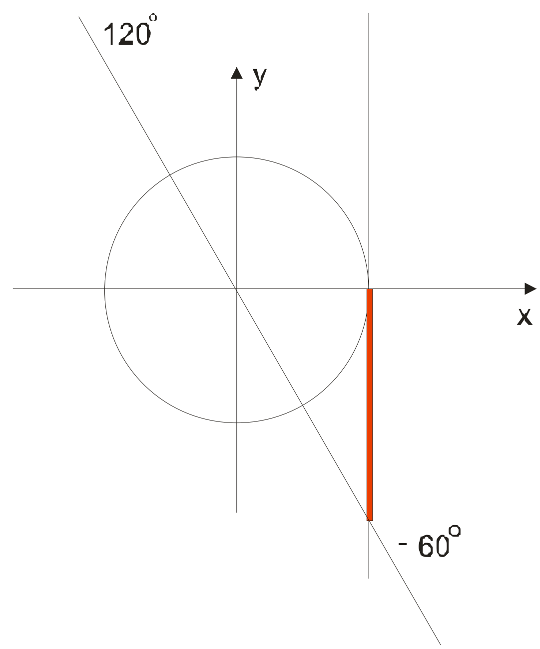
$$z^6 = 2^6(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$

$$z^6 = 64(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 32(1 + i\sqrt{3})$$

b) $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

Here we first must move complex number in the trigonometric form.

$$\left.\begin{aligned} x &= \frac{1}{2} \\ y &= -\frac{\sqrt{3}}{2} \end{aligned}\right\} \Rightarrow r = \sqrt{(\frac{1}{2})^2 + (-\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$




$$tg\varphi = \frac{y}{x}$$

$$tg\varphi = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} \Rightarrow tg\varphi = -\sqrt{3} \Rightarrow \varphi = -60^o = -\frac{\pi}{3}$$

$$z = r(\cos\varphi + i\sin\varphi)$$

$$z = 1(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3}))$$

$$z = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}$$

$$z^{20} = \cos\frac{20\pi}{3} - i\sin\frac{20\pi}{3} \rightarrow \frac{20\pi}{3} = \frac{18\pi}{3} + \frac{2\pi}{3} = \frac{2\pi}{3}$$

$$z^{20} = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$$

$$z^{20} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$z^{20} = -\frac{1}{2}(1 + i\sqrt{3})$$

## Root of complex numbers

If we have $z = r(\cos\varphi + i\sin\varphi)$ then $$\boxed{\sqrt[n]{z} = w = \sqrt[n]{r}(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n})} \quad k = 0,1,2...n-1$$

All values n- the root of z, is on circle with radius $\sqrt[n]{r}$.

**Examples:**

Calculate:

a) $\sqrt[3]{i}$

b) $\sqrt[6]{-1}$

**Solution:**

a) As we already see:

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

Then : $$\sqrt[3]{i} = \cos\frac{\frac{\pi}{2}+2k\pi}{3} + i\sin\frac{\frac{\pi}{2}+2k\pi}{3} \qquad k = 0,1,2$$

For k=0

$$w_o = \cos\frac{\frac{\pi}{2}+0}{3} + i\sin\frac{\frac{\pi}{2}+0}{3}$$

$$w_o = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$$

$$w_o = \frac{\sqrt{3}}{2} + i\frac{1}{2}$$

For k=1

$$w_1 = \cos\frac{\frac{\pi}{2}+2\pi}{3} + i\sin\frac{\frac{\pi}{2}+2\pi}{3}$$

$$w_1 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$

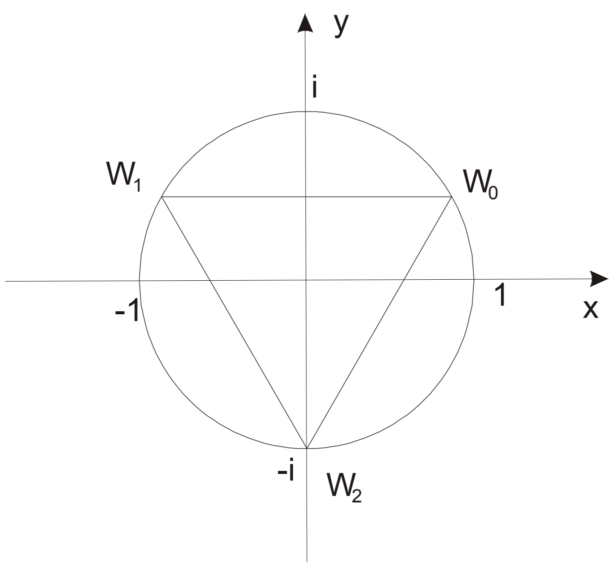
$$w_1 = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$$

For k=2

$$w_2 = \cos\frac{\frac{\pi}{2}+4\pi}{3} + i\sin\frac{\frac{\pi}{2}+4\pi}{3}$$

$$w_2 = \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}$$

$$w_2 = -i$$




b) $-1 = \cos\pi + i\sin\pi \qquad r = 1, \varphi = \pi$

$$\sqrt[6]{-1} = \cos\frac{\pi+2k\pi}{6} + i\sin\frac{\pi+2k\pi}{6}$$

$$k = 0,1,2,3,4,5$$

For k=0

$$w_o = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$$

For k=1

$$w_1 = \cos\frac{\pi+2\pi}{6} + i\sin\frac{\pi+2\pi}{6}$$

$$w_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$

For k=2

$$w_2 = \cos\frac{\pi+4\pi}{6} + i\sin\frac{\pi+4\pi}{6}$$

$$w_2 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}$$

For k=3

$$w_3 = \cos\frac{\pi+6\pi}{6} + i\sin\frac{\pi+6\pi}{6}$$

$$w_3 = \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6} = -\frac{\sqrt{3}}{2} - i\frac{1}{2}$$

For k=4

$$w_4 = \cos\frac{\pi + 8\pi}{6} + i\sin\frac{\pi + 8\pi}{6}$$

$$w_4 = \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6} = -i$$

For k=5

$$w_5 = \cos\frac{\pi + 10\pi}{6} + i\sin\frac{\pi + 10\pi}{6}$$

$$w_5 = \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6} = \frac{\sqrt{3}}{2} - i\frac{1}{2}$$